Some experimental approaches to Diophantine approximation problems

Keith Briggs

Keith.Briggs@bt.com

`research.btexact.com/teralab/keithbriggs.html`

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Aim

- ⋆ To generalize some classical ideas from continued fractions to the problem of *simultaneous approximation* of sets of irrationals by rationals with common denominator
- ⋆ To understand *periodicity* questions related to approximation in number fields
- ⋆ To construct *badly approximable sets*
- ⋆ To study *statistics of blocks of digits* of continued fractions
- ⋆ To do all of the above by *computer experimentation*, when other approaches have failed

Continued fractions

- *(Simple, regular) continued fractions* are symbolic dynamics of the Gauss map:

$$g(x) = 1/x - \lfloor 1/x \rfloor \quad \text{for} \quad x \in (0,1]$$

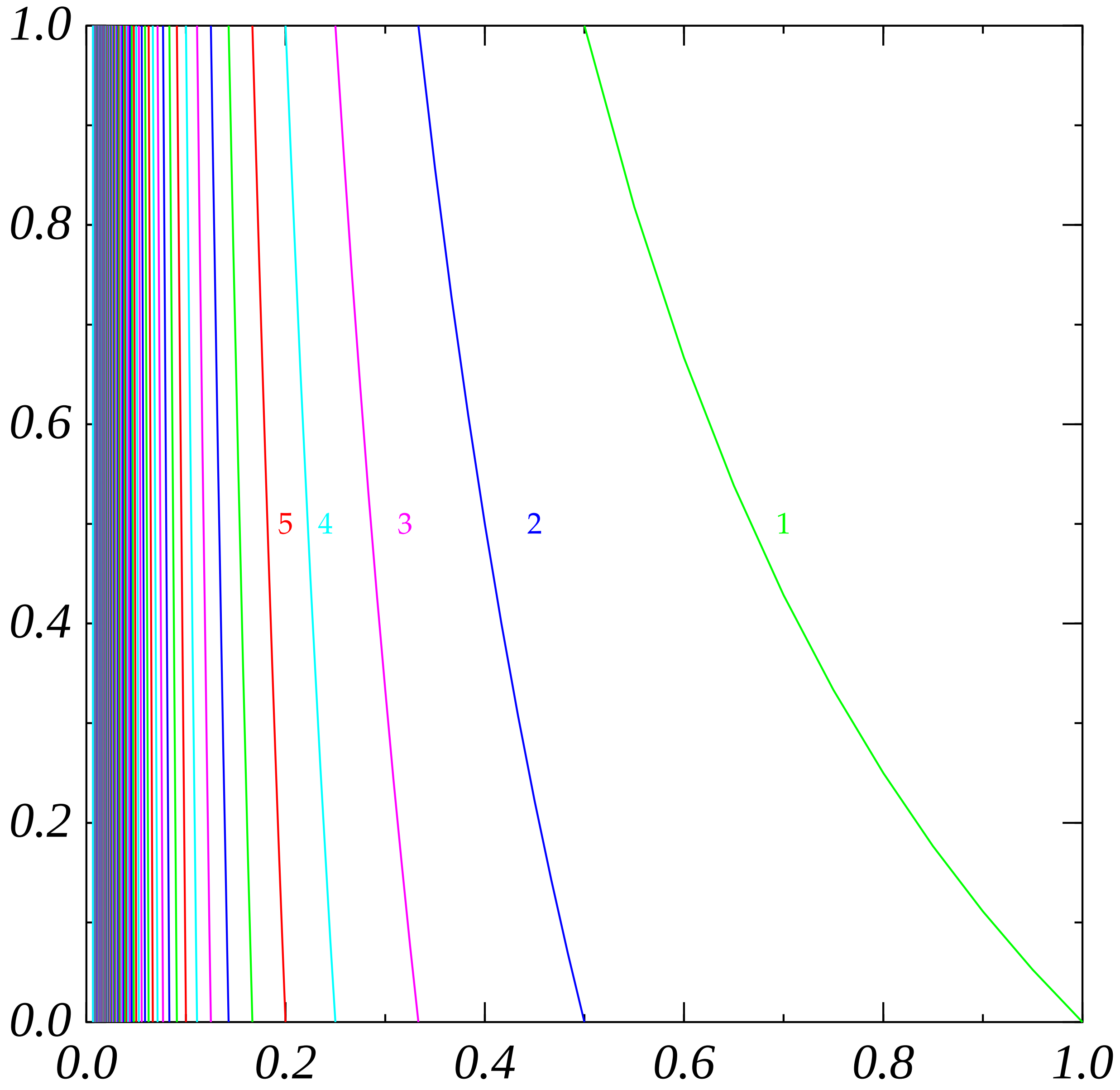
- The *partial quotient* ('digit') $x_k = \lfloor 1/g^{<k-1>}(x) \rfloor$ ($x_k \in \{1,2,3,\dots\}$) is output at the kth iteration

- We write $x = [x_1, x_2, x_3, \dots] \equiv 1/(x_1 + 1/(x_2 + 1/(x_3 + \dots)))$

- The continued fraction is *finite* iff x is rational

- The continued fraction is *eventually periodic* iff x is a quadratic irrational

Gauss map



Approximation properties of continued fractions

- If (for $k = 1, 2, 3, \ldots$)

$$\begin{bmatrix} p_{k-1} & q_{k-1} \\ p_k & q_k \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & x_k \end{bmatrix} \begin{bmatrix} p_{k-2} & q_{k-2} \\ p_{k-1} & q_{k-1} \end{bmatrix}$$

$$\begin{bmatrix} p_{-1} & q_{-1} \\ p_0 & q_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  then $\frac{p_k}{q_k}$ is *precisely* the sequence of best approximants to x

- That is, $|q_K x - p_K| < |q_k x - p_k| \quad \forall k < K$
- In particular, $\frac{1}{(x_{k+1}+2)q_k} < |q_k x - p_k| < \frac{1}{x_{k+1} q_k}$
- Note for small denominator applications: if $\lambda = \exp(2\pi i \alpha)$, then $\forall q$ we have $4|\alpha q - \lfloor q\alpha \rceil| < |\lambda^q - 1| < 2\pi |q\alpha - \lfloor q\alpha \rceil|$

Diophantine approximation in one dimension

- ★ In one dimension, we measure the goodness of approximation of the rational number p/q to α by $c(\alpha,p,q) \equiv q|q\alpha - p|$
- ★ For each irrational α there are infinitely many rationals p/q such that $|\alpha - p/q| < 1/q^2$; that is, $c(\alpha,p,q) < 1$
- ★ The *approximation constant* of α is $c(\alpha) \equiv \liminf_{q\to\infty} c(\alpha, \lfloor q\alpha \rceil, q)$
- ★ Introducing the notation $\wr\alpha\int$ for the distance from α to the nearest integer, we have $c(\alpha) \equiv \liminf_{q\to\infty} q \wr q\alpha \int$
- ★ α is said to be *badly approximable* if $c(\alpha) > 0$
- ★ The *one-dimensional diophantine approximation constant* is $c_1 \equiv \limsup_{\alpha\in\mathbb{R}} c(\alpha)$, and this is known to have the value $1/\sqrt{5}$, attained for example at the golden ratio $\alpha = (\sqrt{5}-1)/2$

Diophantine approximation in two dimensions

In the two-dimensional case, we wish to simultaneously approximate a pair of irrationals by a pair of rationals with common denominator:

- The closeness of approximation is measured by the ***maximum*** error of the two components
- We thus extend the meaning of the symbol $\wr\cdot\wr$ by $\wr\boldsymbol{\alpha}\wr \equiv \min_{\boldsymbol{p}\in\mathbb{Z}^2} \max(|\,q\alpha_1 - p_1\,|, |\,q\alpha_2 - p_2\,|)$
- For $q \in \mathbb{Z},\ \boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \backslash \mathbb{Q}^2$, let

$$c(\boldsymbol{\alpha}, q) \;=\; q\,\wr q\boldsymbol{\alpha}\wr^2, \quad c(\boldsymbol{\alpha}) = \liminf_{q\to\infty} c(\boldsymbol{\alpha}, q)$$

- The ***two-dimensional sup-norm simultaneous diophantine approximation constant*** is then $c_2 = \sup_{\boldsymbol{\alpha}} c(\boldsymbol{\alpha})$
- The value of c_2 is unknown, but bounds are known: $2/7 < c_2 < 64/169$

Algebraic number fields

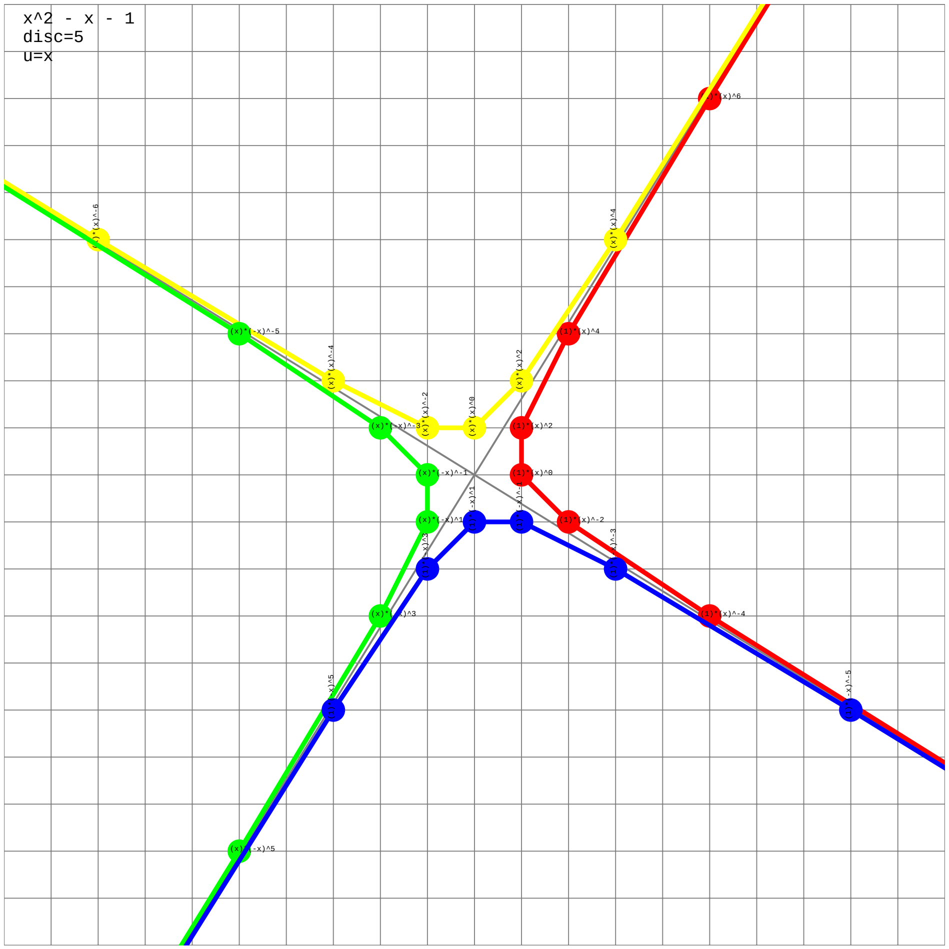
- *K is a number field $\Leftrightarrow \mathbb{Q} \subset K$ and K is a finite-dimensional vector space over $\mathbb{Q}$*
- *if $K = \mathbb{Q}(\theta)$, θ is called a primitive element*
- *The conjugates of θ are the roots θ_i of the minimal polynomial of θ. These define n embeddings of K in $\mathbb{C}$*
- *Each K has a discriminant which is a squared multiple of the discriminant of the minimal polynomial of θ*
- *The norm Nx of an element x is the product of x with its conjugates*
- *A unit is an element of norm ± 1. The set of units $U(K)$ forms a multiplicative group*
- *Dirichlet's theorem: there exists a set of units $\{u_1, \ldots, u_r\}$ such that every unit u can be expressed as $u = \zeta u_1^{n_1} \cdots u_r^{n_r}$, where ζ is a root of unity*
- *Such a set is called a set of fundamental units*
- *In other words, $U(K)$ is isomorphic to a product of a cyclic group and an additive Abelian group*

Algebraic number fields cotd.

- $x \in K$ *is an* ***algebraic integer*** *if it is the root of a monic polynomial* $f \in \mathbb{Z}[x]$
- *The set of algebraic integers in* K *form a ring* $\mathbb{Z}_K$
- *A* $\mathbb{Z}$*-basis of* $\mathbb{Z}_K$ *considered as a* Z*-module is called an* ***integral basis***

polynomial	field	d	integral basis	fund. units
x^2-x-1	$\mathbb{Q}(\sqrt{5})$	5	$[1, x]$	$\{x\}$
x^2-5	$\mathbb{Q}(\sqrt{5})$	5	$[1, (x+1)/2]$	$\{(x+1)/2\}$
x^2-2	$\mathbb{Q}(\sqrt{2})$	8	$[1, x]$	$\{x-1\}$
x^2-3	$\mathbb{Q}(\sqrt{3})$	12	$[1, x]$	$\{x-2\}$
x^3+2x^2-x-1	$\mathbb{Q}\left(2\cos(\frac{2\pi}{7})\right)$	49	$[1, x, x^2]$	$\{x^2-1, x+1\}$
x^3-x-1	$\mathbb{Q}\,(?)$	-23	$[1, x, x^2-1]$	$\{x\}$

Klein polygons - $\mathbb{Q}(\sqrt{5})$



K=(Field of $x^2 - x - 1$) = $\mathbb{Q}(\sqrt{5})$

Fundamental unit: $u = x$

Norm of f.u.: $Nu = -1$

Integral basis: $B = \{1, x\}$

$(\sqrt{5} - 1)/2 = [\overline{1}]$

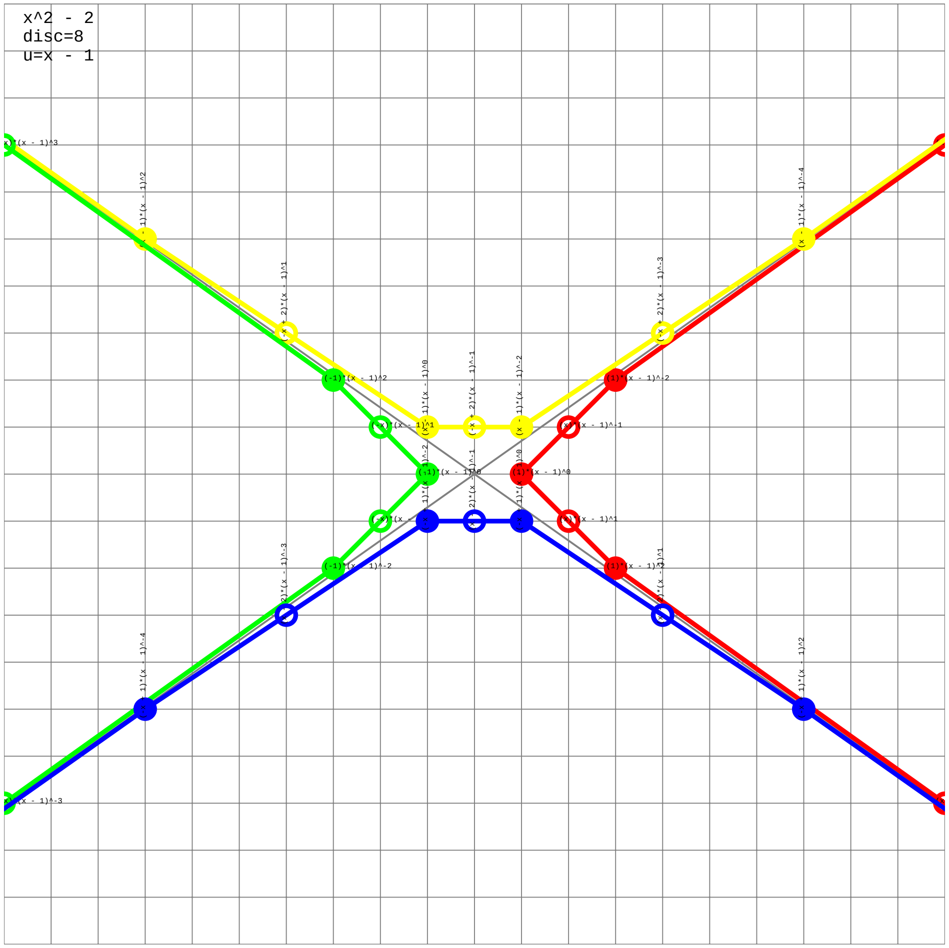
$\boldsymbol{u^{2\mathbb{Z}}}$

$\boldsymbol{u^{2\mathbb{Z}-1}}$

$\boldsymbol{-u^{2\mathbb{Z}}}$

$\boldsymbol{-u^{2\mathbb{Z}-1}}$

Klein polygons - $\mathbb{Q}(\sqrt{2})$



K=(Field of x^2-2) = $\mathbb{Q}(\sqrt{2})$
Fundamental unit: $u=x-1$
Norm of f.u.: $Nu=-1$
Integral basis: $B=\{1,x\}$
$\sqrt{2}=[1,\overline{2}]$
$\boldsymbol{u^{2\mathbb{Z}}}$
$\boldsymbol{u^{2\mathbb{Z}-1}}$
$\boldsymbol{-u^{2\mathbb{Z}}}$
$\boldsymbol{-u^{2\mathbb{Z}-1}}$

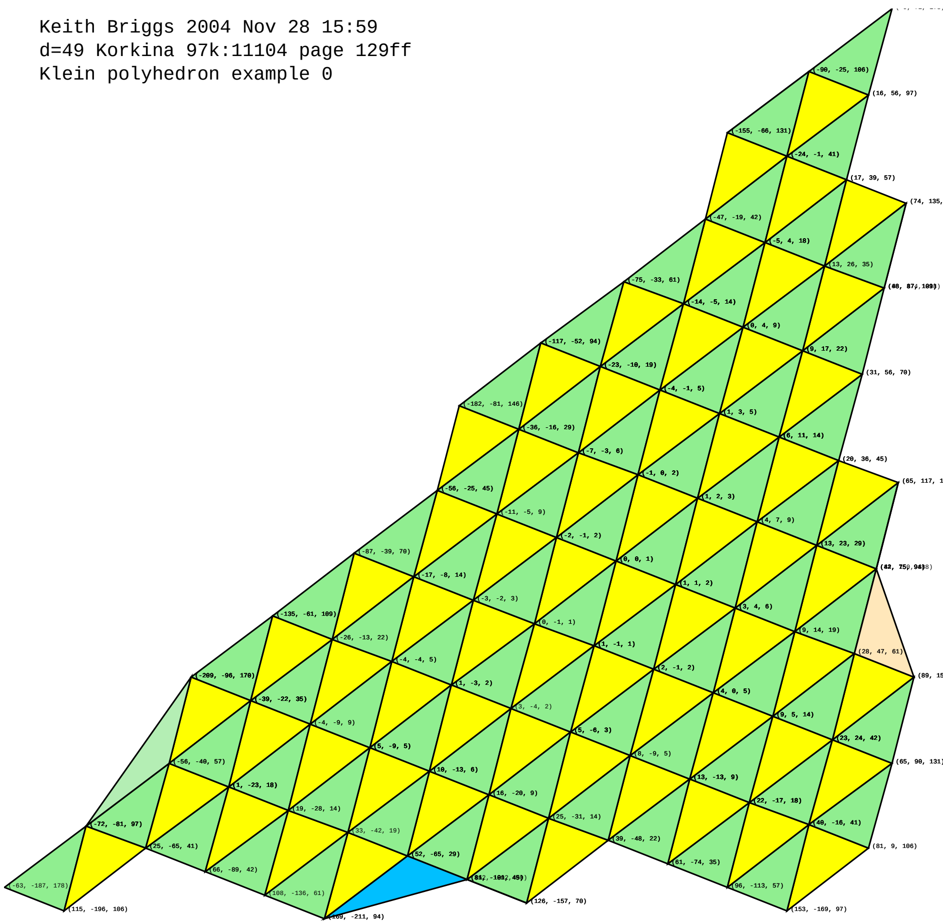
Klein polyhedra

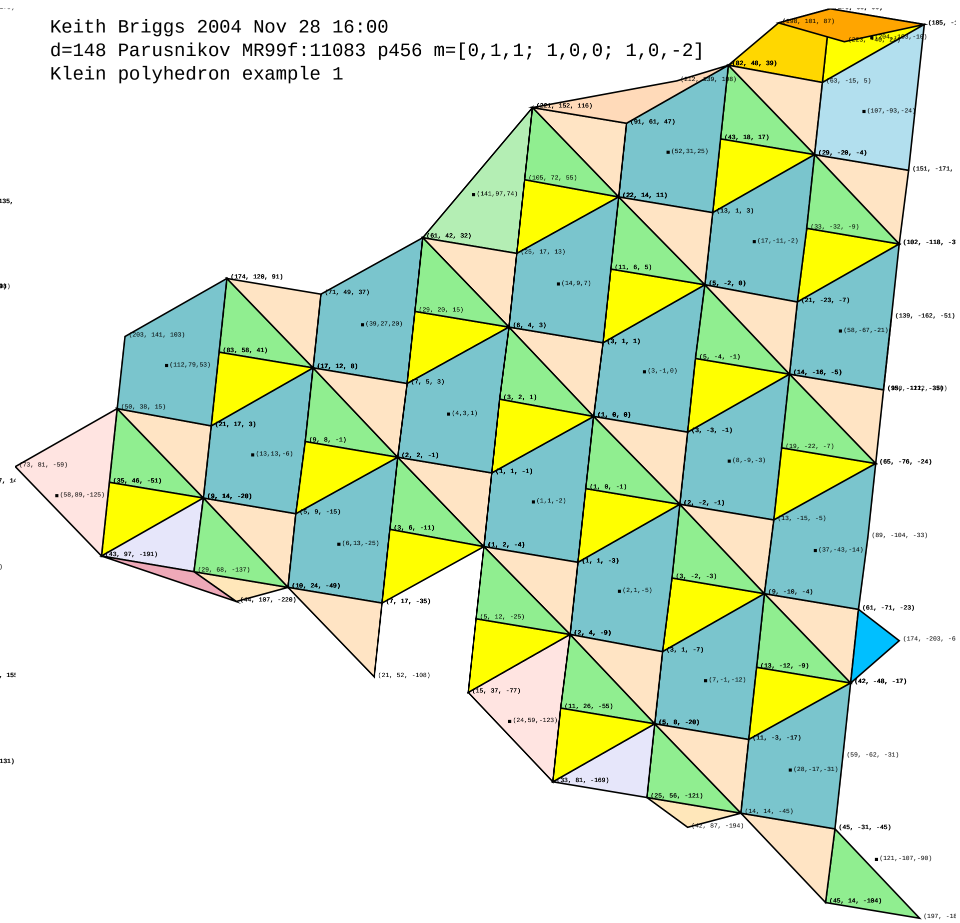
- We work in $\mathbb{R}^3$, and in the lattice $\mathbb{Z}^3$ embedded in it
- Consider three planes with (not necessarily unit) normals α_i $(i = 1, 2, 3)$ through the origin. Consider the octant defined by $\alpha_i(x) > 0$
- Now form the convex hull of the points of $\mathbb{Z}^3$ (excluding the origin) contained in this octant. This is the ***Klein polyhedron*** of the cubic form $\alpha(x) = \alpha_1(x)\alpha_2(x)\alpha_3(x)$
- To visualize a Klein polyhedron, we take a finite piece near the origin and flatten the piece onto a plane. More precisely, for each vertex point x in the convex hull, we let $y = |\alpha(x)|^{-1/3}x$ and plot (z_1, z_2), where

$$z = (\log|\alpha_1(y)|, \log|\alpha_2(y)|, \log|\alpha_3(y)|)$$

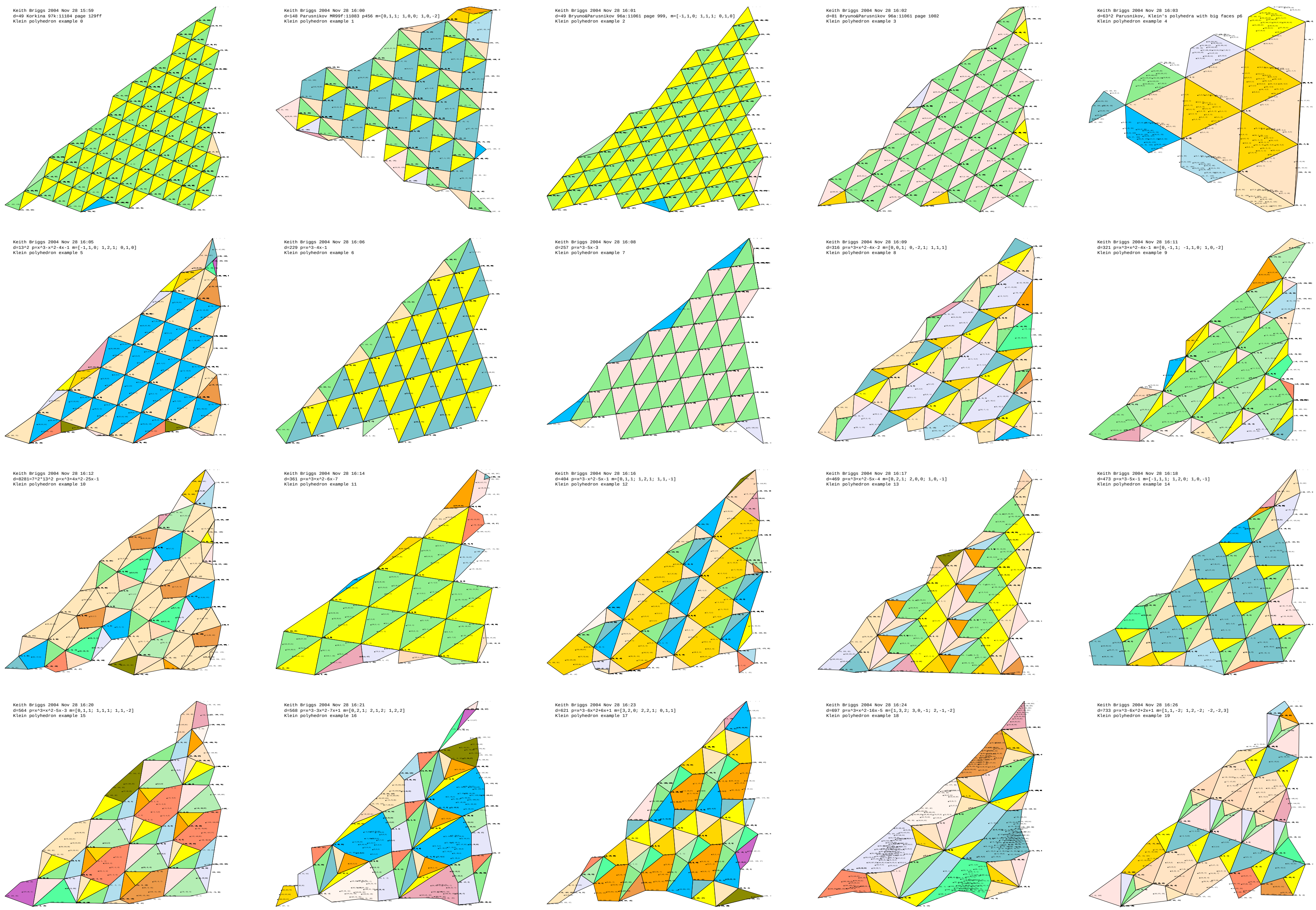
- Thus $z_1+z_2+z_3 = 0$, so by plotting any two components we have full information
- The main point of interest is that the patterns are periodic iff the planes are related to a totally real cubic number field

Klein polyhedra examples - $d = 49$ and $d = 148$





More Klein polyhedra examples



The Furtwängler algorithm

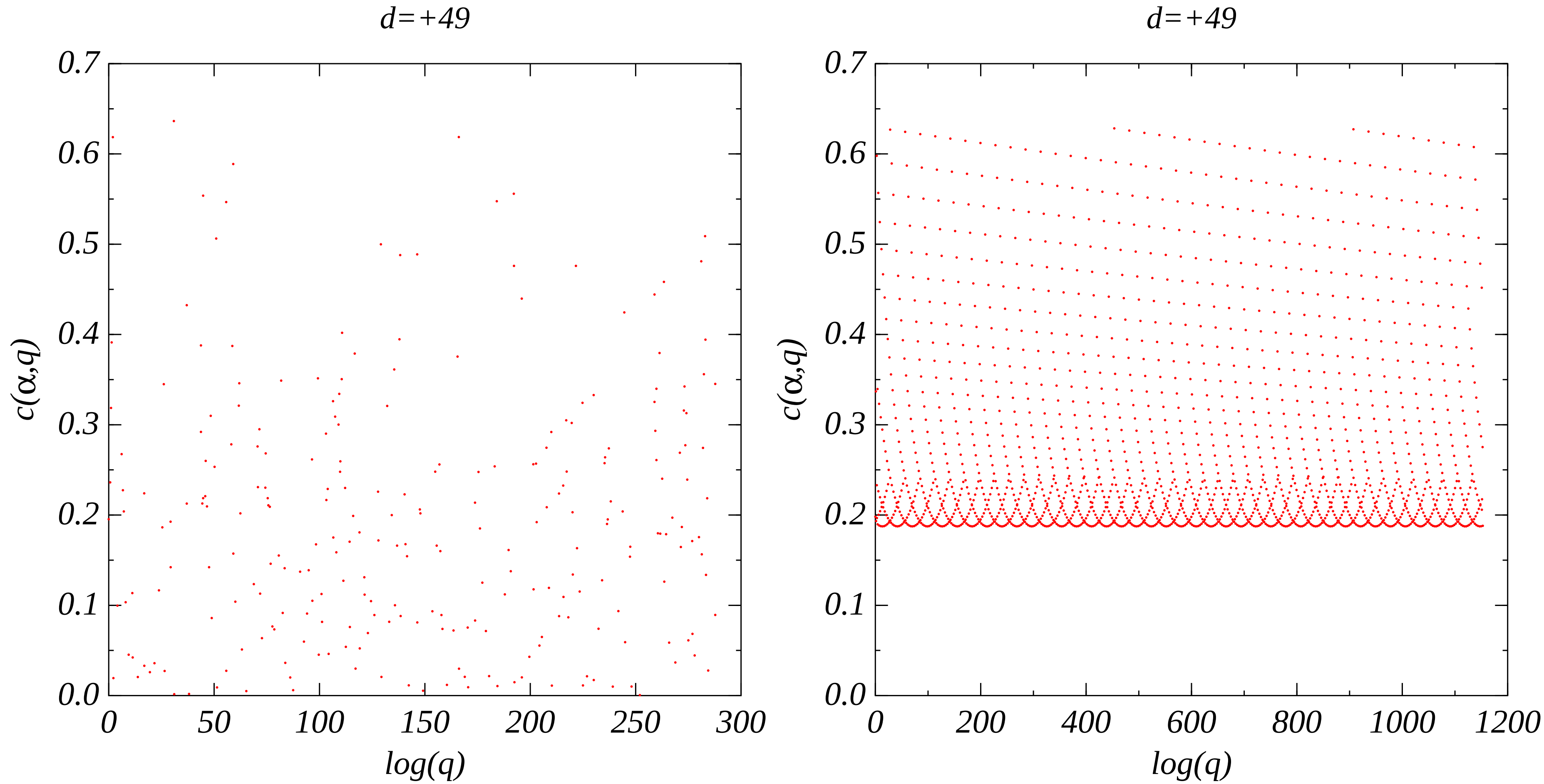
- The aim of the Furtwängler algorithm [Fur26, Fur28] is to find all best sup-norm simultaneous rational approximations to a given irrational pair $(\alpha_1, \alpha_2) \in \mathbb{R}^2 \backslash \mathbb{Q}^2$
- $(Q, P_1, P_2) \in \mathbb{Z}^3$ is called a *best approximation triple* if $\forall q < Q$

$$\max(|q\alpha_1 - p_1|, |q\alpha_2 - p_2|) > \max(|Q\alpha_1 - P_1|, |Q\alpha_2 - P_2|)$$

  - *The algorithm works by keeping an approximation matrix A whose rows we label P, Q, R. A typical row contains integers (p_1, p_2, q) corresponding to an approximant $(p_1/q, p_2/q)$, which need not be a best approximant. A step of the main loop of the algorithm consists of finding a new row S which will replace one of P, Q or R, and a reordering of the rows. The new row S is always an integer linear combination of P, Q and R*
  - *Choosing S involves considering several possibilities, and the final choice is such that no best approximant will be missed, though the number of iterations between best approximations may be arbitrarily large*
  - *there are five different unimodular update matrices B (with $a, b \in \mathbb{Z}$):*

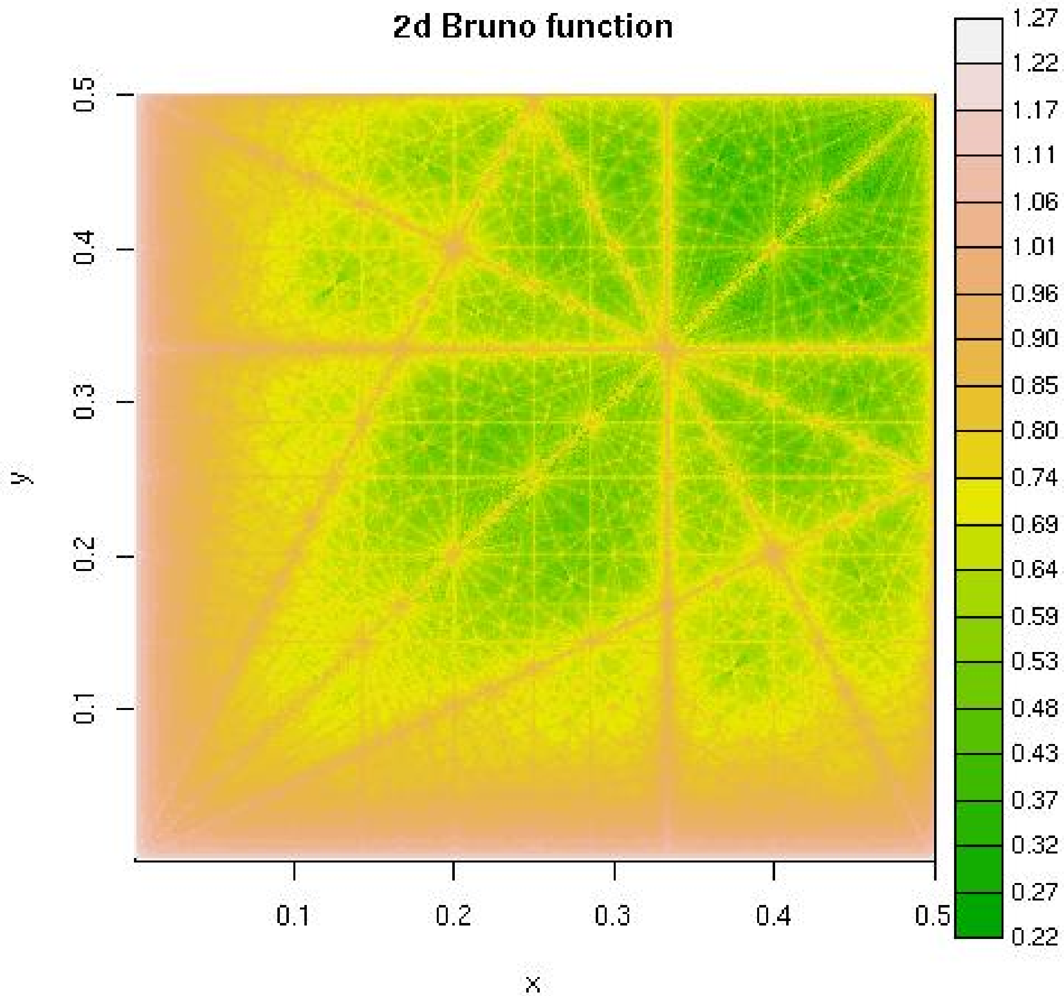
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ a & b & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a & b & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The Furtwängler algorithm - behaviour



$c(\boldsymbol{\alpha}, q)$ at best approximation denominators q. Left: $\boldsymbol{\alpha}=$ a 'random' pair of irrationals. Right: $\boldsymbol{\alpha} = (4t^2-1, 2t-1), t = \cos(2\pi/7)$.

The 2-dimensional Bruno function



$B(\boldsymbol{\alpha}) \equiv \sum_{i=0}^{\infty} \log{(q_{i+1})}/q_i$ where $\{q_0, q_1, \ldots\}$ are the BSADs of $\boldsymbol{\alpha} = (x, y)$

The worst approximable pair

- In [Bri03], I used some theorems of Cusick to explicitly construct some provably badly approximable pairs

- The worst pair found was

$$\begin{aligned} \alpha_1 &\approx 0.484873957288933295198967824780619062115945633665761 \\ \alpha_2 &\approx 0.540492503500466747825742853957575236742411192672356 \end{aligned}$$

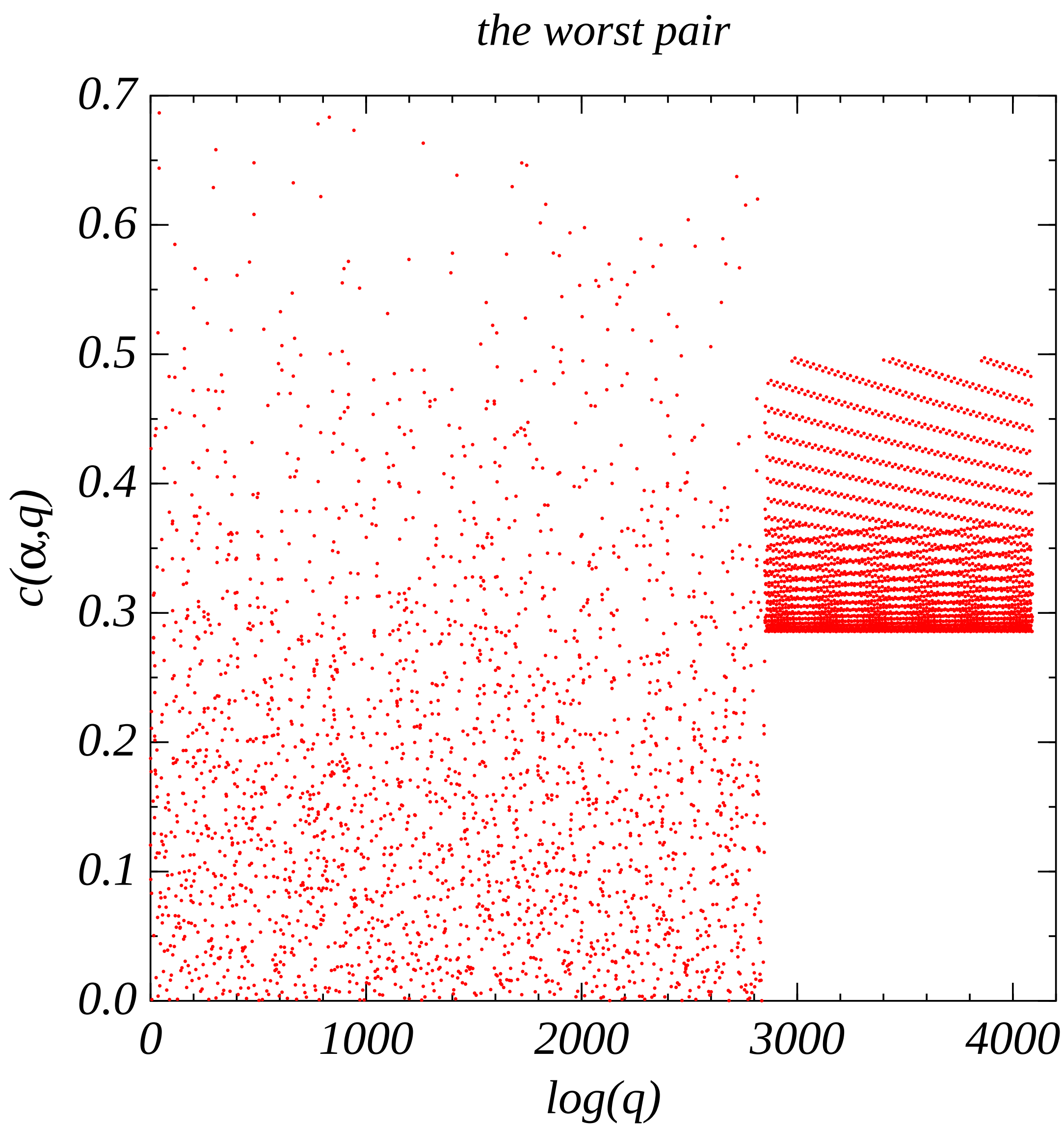
which has $c_2(\boldsymbol{\alpha}) > 0.2857082$

- To fully specify this pair would take several thousand bits

- The method depends on finding sequences in the continued fraction of $2\cos(2\pi/7)$ of the form $[\ldots, n_1, 1, 1, n_2, \ldots]$ with n_1, n_2 large

- It is not known whether n_1, n_2 become arbitrarily large. If so, $c_2 = 2/7$ can be obtained in this field

The worst approximable pair in Furtwängler's algorithm



The $\mathbb{Z}^4$ map for partial quotients of $2\cos(2\pi/7)$

- By a method of Lagrange, the partial quotients of $2\cos(2\pi/7)$ are given by the output of this map on $\mathbb{Z}^4$:
- **Algorithm**:
  - **Initialize:**

    $$z \leftarrow [-1, -2, 1, 1] = [z_0, z_1, z_2, z_3]$$

  - **repeat:**

    $w \leftarrow 1$
    for j in 2,1,0: z_j += z_{j+1}
    for j in 2,1: z_j += z_{j+1}
    for j in 2: z_j += z_{j+1}
    if $z_0+z_1+z_2+z_3 < 0$: w += 1
    else: output w; $z \leftarrow [-z_3, -z_2, -z_1, -z_0]$

Ergodic properties of continued fractions

For almost all irrational x, ordinary 1d continued fractions have these properties as a consequence of the invariant measure of the Gauss map being $\log_2(1+x)$:

- ★ the digit i occurs with relative frequency $\mu(i) \equiv \log_2 \left[\frac{(i+1)^2}{i(i+2)}\right]$
- ★ $\lim_{k\to\infty}(x_1 x_2 x_3 \ldots x_k)^{1/k} = 2.68545\ldots$
- ★ The *denominator growth rate* is $g_1 \equiv \lim_{k\to\infty} {q_k}^{1/k} = \exp\left(\pi^2/(12\log 2)\right) = 3.27582\ldots$

Lagarias has proven that g_2, the growth rate of 2d BSADs is bounded by $g_2 > \sqrt{\frac{1+\sqrt{5}}{2}} = 1.27202\ldots$

- ★ Using Furtwängler's algorithm, I estimated that the average g_2 is really about 3.07
- ★ I found the smallest g_2 in the field of discriminant -23, where $g_2 \approx 1.563$

Statistics of blocks for 1d continued fractions

- I will look at occurrences of *finite blocks* of digits $i=(i_1,i_2,\ldots,i_m), i_j\geqslant 1$
- [IK02] gives a formula for relative frequency of the m-block i which holds $\forall\epsilon>0$ as $n\to\infty$ for almost all irrationals:

$$\text{card}\{\kappa:(x_\kappa,\ldots,x_{\kappa+m-1})=i\ ,1\leqslant\kappa\leqslant n\}/n=\log_2\left[\frac{1+v(i)}{1+u(i)}\right]+o\left(n^{-1/2}\log^{(3+\epsilon)/2}(n)\right)$$

where (with $[i]=p_m/q_m$ for the m-block i)

$$u(i)=\begin{cases}(p_m+p_{m-1})/(q_m+q_{m-1}) & \text{if } m \text{ is odd}\\ p_m/q_m & \text{if } m \text{ is even}\end{cases}$$

$$v(i)=\begin{cases}p_m/q_m & \text{if } m \text{ is odd}\\ (p_m+p_{m-1})/(q_m+q_{m-1}) & \text{if } m \text{ is even}\end{cases}$$

Numerical values for the frequencies

For 2-blocks:

	1	2	3	4	5	6
1	0.15200	0.07038	0.04064	0.02647	0.01861	0.01380
2	0.07038	0.02914	0.01594	0.01005	0.00691	0.00505
3	0.04064	0.01594	0.00851	0.00529	0.00361	0.00262
4	0.02647	0.01005	0.00529	0.00326	0.00221	0.00160
5	0.01861	0.00691	0.00361	0.00221	0.00150	0.00108
6	0.01380	0.00505	0.00262	0.00160	0.00108	0.00078

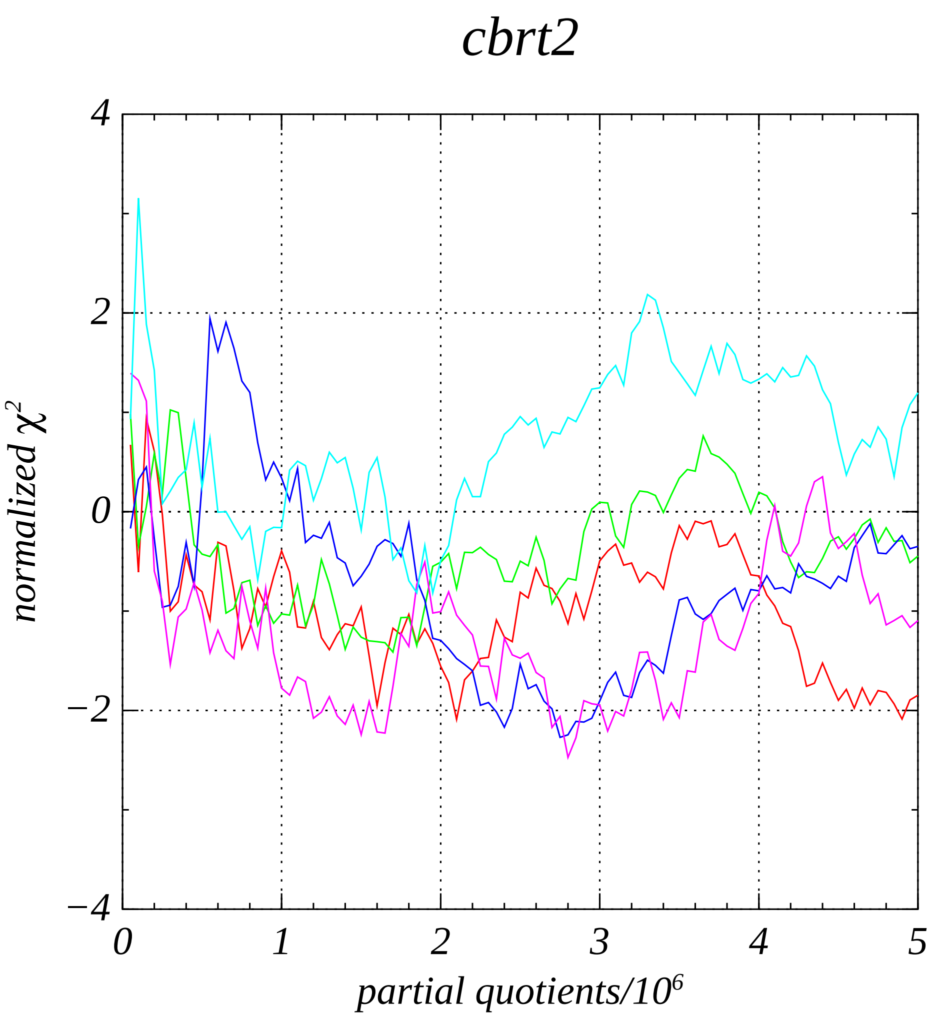
Explicit examples of abnormal numbers

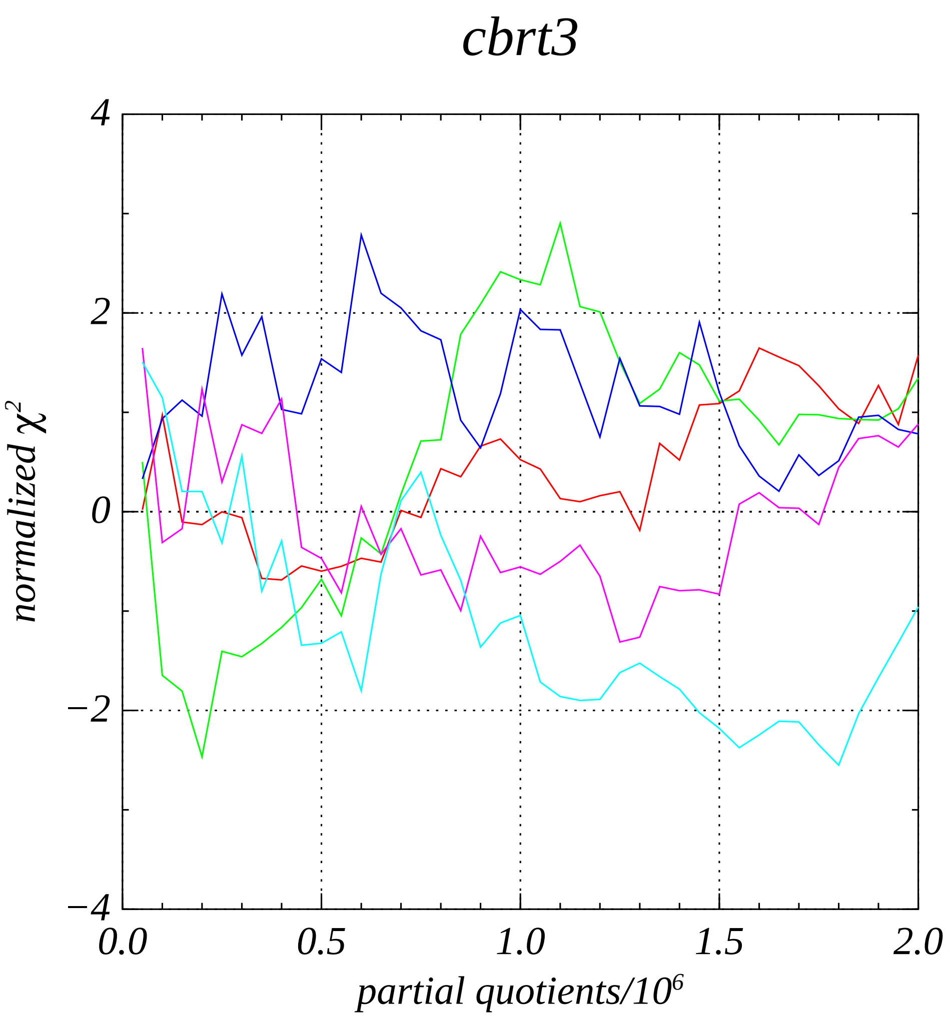
- all quadratic irrationals, e.g. $2^{1/2} = 1+[2, 2, 2, 2, \ldots]$
- $I_1(2)/I_0(2) = [1, 2, 3, 4, \ldots]$ (ratio of modified Bessel functions)
- $I_{1+a/d}(2/d)/I_{a/d}(2/d) = [a+d, a+2d, a+3d, \ldots]$
- $\tanh(1) = [1, 3, 5, 7, \ldots]$
- $\exp(1/n) = [1, n-1, 1, 1, 3n-1, 1, 1, 5n-1, \ldots];\ n = 1, 2, 3\ldots$
- $\exp(2) = 7+[2, 1, 1, 3, 18, 5, 1, 1, 6, 30, 8, 1, 1, 9, 42, 11, 1, 1, \ldots]$
- $\exp(2/(2n+1));\ n = 1, 2, 3\ldots$
- $\sum_{k=1}^{\infty} 2^{-\lfloor k\phi \rfloor} = [2^0, 2^1, 2^1, 2^3, 2^5, 2^8, 2^{13}, \ldots];\ \phi = (\sqrt{5}-1)/2$

Method

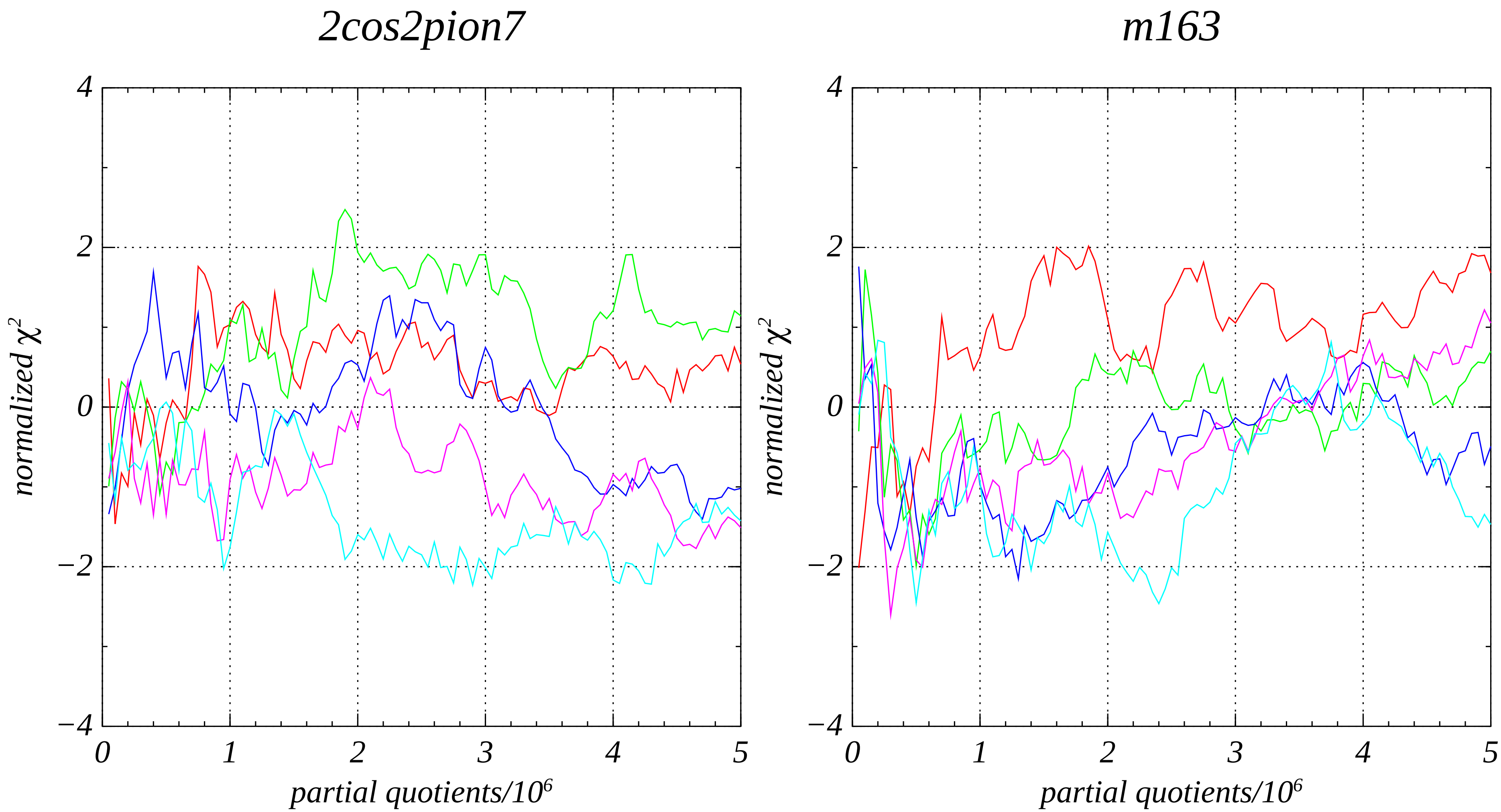
- I calculated a few million digits for several cubic irrationals and a few other irrationals
- I counted exactly the observed frequency of all blocks of lengths 1,2,3,4, and 5
- I calculated a Pearson χ^2 test statistic which measures the deviation of the observed frequencies from the expected frequencies
- Because the number of degrees of freedom ν is so large (typically several thousand), a normal approximation is sufficiently accurate. The transformation is $Z \equiv \sqrt{2\chi^2} - \sqrt{2\nu - 1}$. Under the assumption of normality (of the cf of x!), Z is distributed $N(0,1)$

χ^2 results: $2^{1/3}$ and $3^{1/3}$



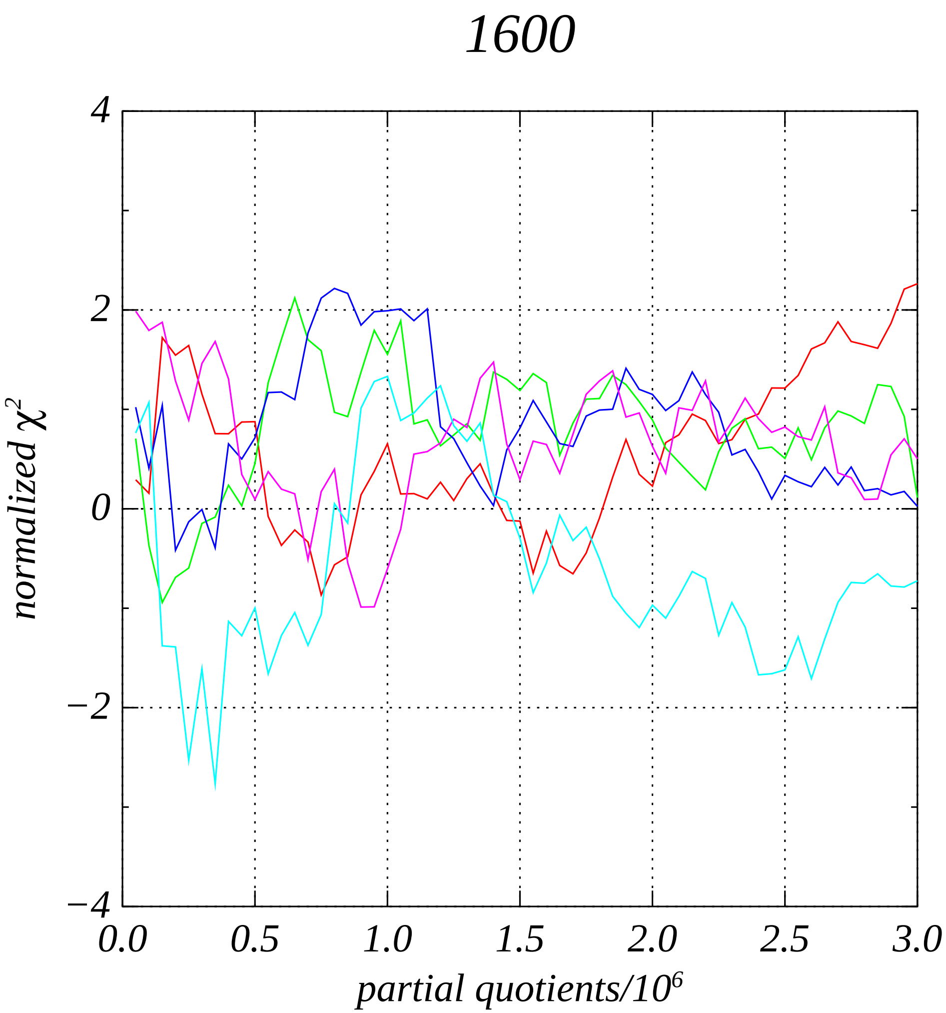


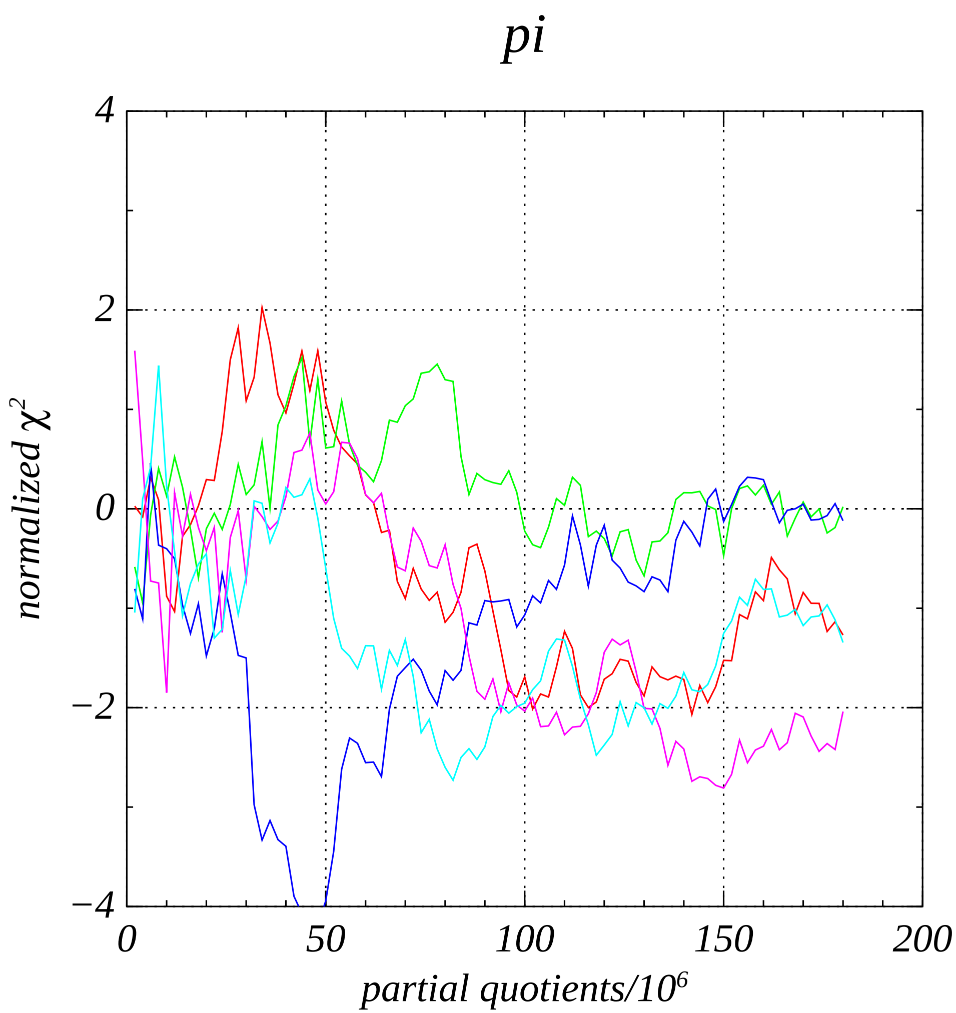
χ^2 results: $2\cos(2\pi/7)$ and largest root of $x^3-8x-10$



(the last example is famous for having several abnormally large digits)

χ^2 results: $(\sqrt{5}-1)/2+\sqrt{2}-1$ and π





Autocorrelation of digits

- We would expect that the autocorrelation function (acf) of any analytic function of the digits that has a finite mean (for example, the log or the reciprocal) would decay like q^k at lag k, where $q \approx -0.303663$ is *Wirsing's constant*
- This is investigated in the following graphs. I plot $\log_{10}$ of the absolute value of the acf as a function of lag. The green line has the Wirsing slope
- In Rockett & Szüsz [RS92], we have the result

$$\text{Pr}\left[x_n = r \;\&\; x_{n+k} = s\right] = \text{Pr}\left[x_n = r\right] \text{Pr}\left[x_{n+k} = s\right] \left(1 + O(q^k)\right)$$

This, however, is too weak to allow explicit statistical tests

acf estimation difficulties

★ For the AR(1) process $x(t+1) = \alpha x(t) + \epsilon$, $|\alpha| < 1$, the exact acf at lag k is $\rho(k) = \alpha^k$

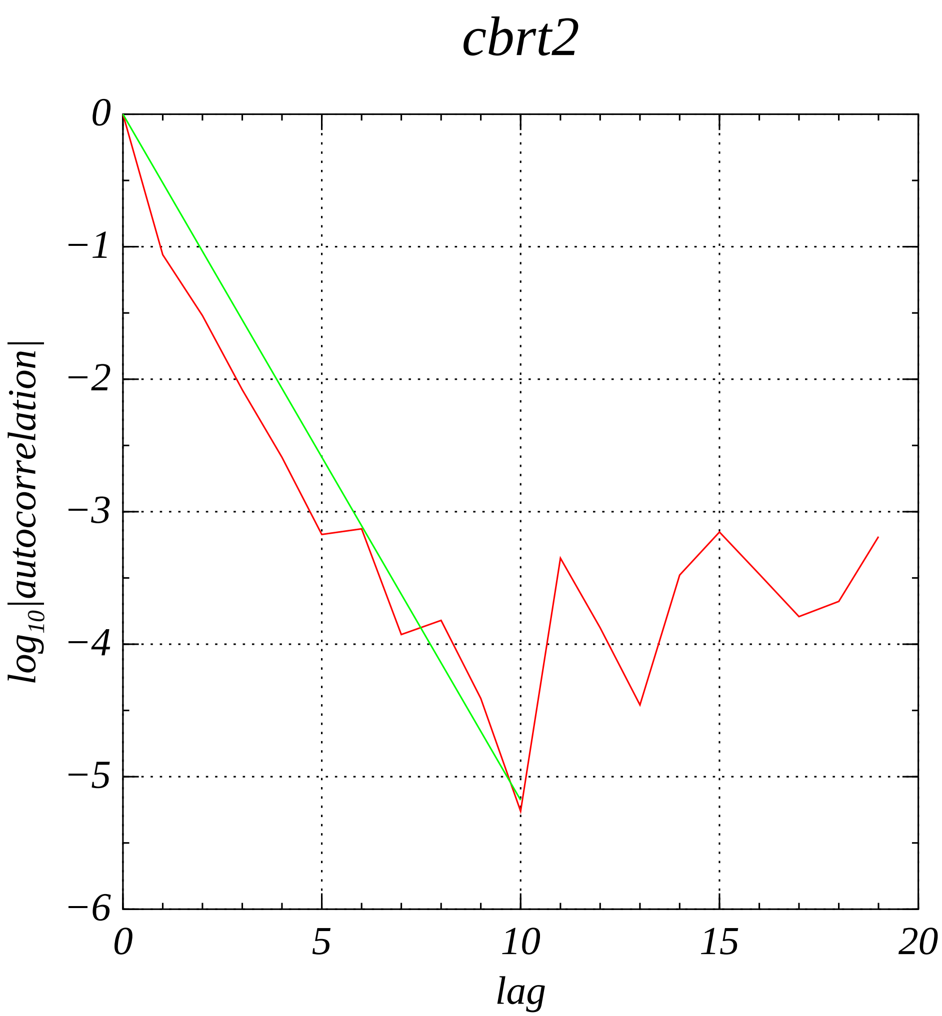
★ But the usual acf estimator r for a sample of size n has variance

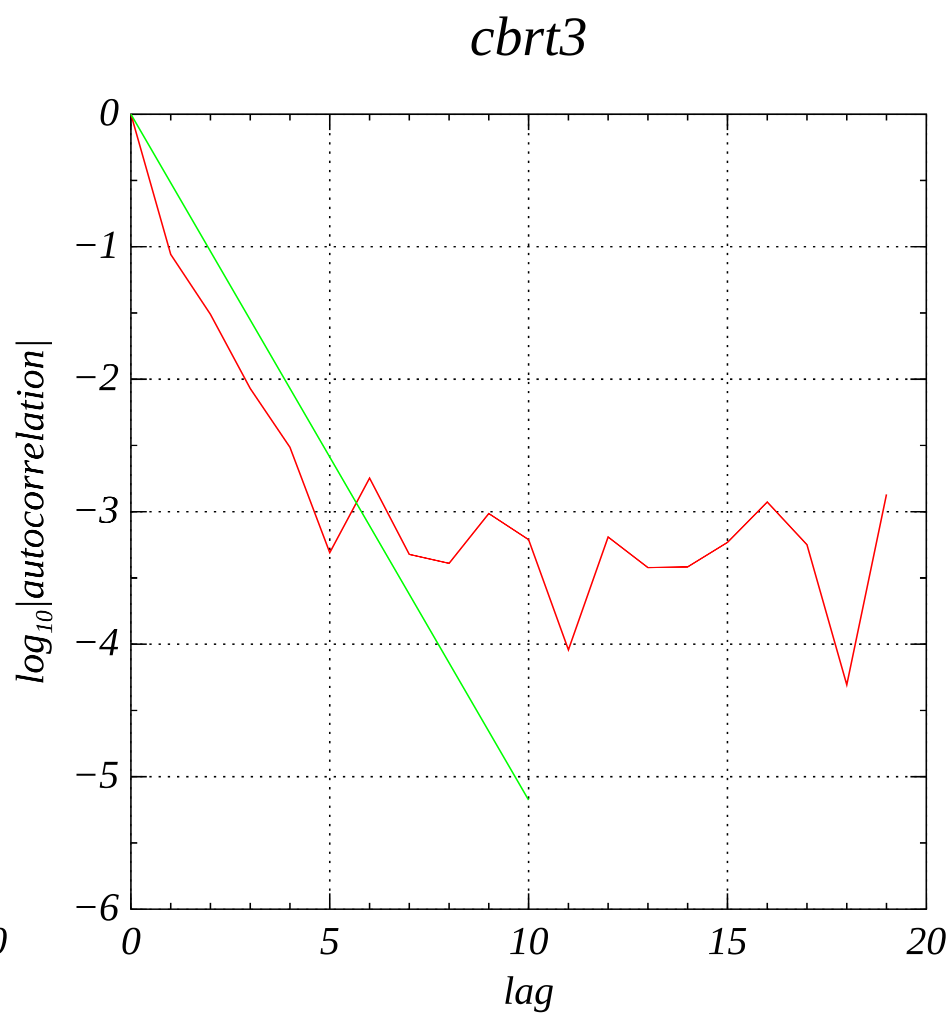
$$\text{var}\left[r_n(k)\right] = \frac{1}{n}\left[\frac{(1+\alpha^2)(1+\alpha^{2k})}{1-\alpha^2} - 2k\alpha^{2k}\right]$$

★ More generally, for a process whose acf decays for large k in the same power-law fashion, we have approximate variance $\text{var}\left[r_n(k)\right] = \frac{1}{n}\left[\frac{1+\alpha^2}{1-\alpha^2}\right]$ for large k

★ I expect my process to conform to this behaviour, and if it does, putting in the numbers gives an estimate of $k = 6$ for the largest k for which the acf estimates are meaningful

autocorrelation of logs of digits: $2^{1/3}$ and $3^{1/3}$





Appendix: history 1

- *In the early 1900s, Minkowski's work on geometry of numbers [Min11b, Min11a] provided a theoretical basis for all subsequent work. Perron, Klein etc. established the metric theory of one-dimensional continued fractions.*
- *In the 1920s, Furtwängler published the first algorithm which aimed to find all the best sup-norm approximants up to a given denominator. This work was largely forgotten [Fur26, Fur28, Bri01].*
- *1950s: Davenport, Cassels.*
- *In 1970, G. Szekeres [Sze70] published his multi-dimensional continued fraction algorithm. It is now known that this does not find all best sup-norm approximants.*
- *In the 1970s there was significant work by Cusick, Adams and Krass (see bibliography) especially on the relation of two-dimensional approximation to cubic number fields.*
- *In 1981, Brentjes completed his thesis, which made major contributions to the field, in particular to two-dimensional Euclidean norm approximation. The published version of the thesis is now a basic reference in the field ([Bre81])*

Appendix continued

- *Around 1985, G. Szekeres published three papers on computer experiments intended to search for the 2 and 3 dimensional approximation constants. This work was not fully rigorous but gave intriguing results which have never been followed up.*

- *[Sch95] studied ergodic properties of various algorithms.*

- *In the 1990s several Russian and French mathematicians developed the concept of Klein polyhedra ([Lac93, BP94, Kor94, Kor95, BP97, Lac98a, Lac98b, Arn98, KS99])*

- *In 1995, Lagarias and Pollington published a clear analysis of the Szekeres multi-dimensional continued fraction algorithm ([LP95])*

- *In 1997, Clarkson ([Cla97]) completed his thesis in which was presented for the first time an algorithm provably finding all best approximants in two dimensions with respect to arbitrary radius and height functions.*

- *In the late 1990s, Khanin and Hardcastle proved some results about an n-dimensional Gauss map [HK00a, HK00b].*

- *A very good summary of the state-of-the-art is [Mos99]. See also [Sch96].*

References

[Arn98] V. I. Arnold. Higher dimensional continued fractions. ***Regular and chaotic dynamics***, 3(3):10–17, 1998. MR 2000h:11012.

[BP94] A. D. Bryuno and V. I. Parusnikov. Klein polyhedra for two cubic Davenport forms. ***Mathematical notes***, 56(3-4):9–27, 1994. Keldysh Institute of the RAS, preprint 48.

[BP97] A. D. Bryuno and V. I. Parusnikov. Comparison of various generalizations of continued fractions. ***Mathematical notes***, 61:278–286, 1997. Keldysh Institute of the RAS, preprint 52.

[Bre81] A. J. Brentjes. ***Multi-dimensional Continued Fraction Algorithms***, volume 145 of ***Mathematical Centre Tracts***. Mathematisch Centrum Amsterdam, 1981. MR 83b:10038.

[Bri01] K. M. Briggs. On the Furtwängler algorithm for simultaneous rational approximation. ***preprint***, ?:?, 2001.

[Bri03] K. M. Briggs. Some explicit badly approximable pairs. ***Journal of Number Theory***, 103:71–76, 2003. doi:10.1016/S0022-314X(03)00104-5.

[Cla97] I. Vaughan L. Clarkson. ***Approximation of Linear Forms by Lattice Points, with applications to signal processing***. PhD thesis, Australian National University, 1997.

[Fur26] Ph. Furtwängler. Über die simultane Approximation von Irrationalzahlen (Erste Mitteilung). ***Math. Annalen***, 96:169–175, 1926.

[Fur28] Ph. Furtwängler. Über die simultane Approximation von Irrationalzahlen (Zweite Mitteilung). ***Math. Annalen***, 99:71–83, 1928.

[HK00a] D. M. Hardcastle and K. Khanin. Almost everywhere strong convergence of multidimensional continued fraction algorithms. Technical Report HPL-BRIMS-00-12, BRIMS, Bristol, UK, 2000.

[HK00b] D. M. Hardcastle and K. Khanin. Continued fractions and the d-dimensional Gauss transform. Technical Report HPL-BRIMS-00-15, BRIMS, Bristol, UK, 2000.

[IK02] M Iosifescu and C Kraaikamp. ***Metrical Theory of Continued Fractions***. Kluwer, 2002.

[Kara] O. N. Karpenkov. On constructing multidimensional periodic continued fractions. http://front.math.ucdavis.edu/math.NT/0411031.

[Karb] O. N. Karpenkov. On examples of two-dimensional periodic continued fractions. http://front.math.ucdavis.edu/math.NT/0411054.

[Kor94] E. Korkina. The periodicity of multidimensional continued fractions. ***C. R. Acad. Sci.***, 319:777–780, 1994. MR 95j:11064.

[Kor95] E. I. Korkina. Two-dimensional continued fractions. The simplest examples. ***Proc. Steklov Institute of Mathematics***, 209:124–144, 1995. MR 97k:11104.

[KS99] M. L. Kontsevich and Yu. M. Suhov. Statistics of Klein polyhedra and multidimensional continued fractions. In ***Pseudoperiodic topology***, volume 197 of ***Am. Math. Soc. Transl.***, pages 9–27. 1999. Dedicated to V. I. Arnold on his 60th anniversary.

[Lac93] Gilles Lachaud. Polyèdre d'Arnol'd et voile d'un cône simplicial: analogues du théorème de Lagrange. ***C. R. Acad. Sci. Paris***, 317:711–716, 1993.

[Lac98a] Gilles Lachaud. Klein polygons and geometric diagrams. In *Contemporary mathematics*, volume 210, pages 365–372. 1998. MR 99a:11086.

[Lac98b] Gilles Lachaud. Sails and Klein polyhedra. In *Contemporary mathematics*, volume 210, pages 373–385. 1998. MR 98k:11094.

[LP95] J. C. Lagarias and Andrew D. Pollington. The continuous diophantine approximation mapping of Szekeres. *J. Austral. Math. Soc.*, A59:148–172, 1995.

[Min11a] H. Minkowski. Zur geometrie der Zahlen. In David Hilbert, editor, *Gesammelte Abhandlungen*, volume 2, pages 43–52. 1911. reprinted Chelsea Pub. Co. 1967.

[Min11b] H. Minkowski. Zur theorie der Kettenbrüche. In David Hilbert, editor, *Gesammelte Abhandlungen*, volume 1, pages 278–292. 1911. Reprinted Chelsea Pub. Co. 1967.

[Mos99] N. G. Moshchevitin. Continued fractions, multidimensional Diophantine approximations and applications. *J. de Théorie des Nombres de Bordeaux*, 11:425–438, 1999. www.emis.de/journals/JTNB/.

[RS92] A. M. Rockett and P. Szüsz. *Continued Fractions*. World Scientific, 1992.

[Sch95] F. Schweiger. *Ergodic theory of fibred systems and metric number theory*. Clarendon Press, Oxford, 1995.

[Sch96] W. W. Schmidt. *Diophantine Approximation*, volume 785 of *Lecture Notes in Mathematics*. Springer-Verlag, first edition, 1996. Second printing.

[Sze70] G. Szekeres. Multidimensional continued fractions. *Annales Universitatis Scientarium Budapestinenses de Rolando Eőtvős Nominatate, sectio mathematica*, 8:113–140, 1970. MR 47 #1753.