Notes on labelled planar graphs

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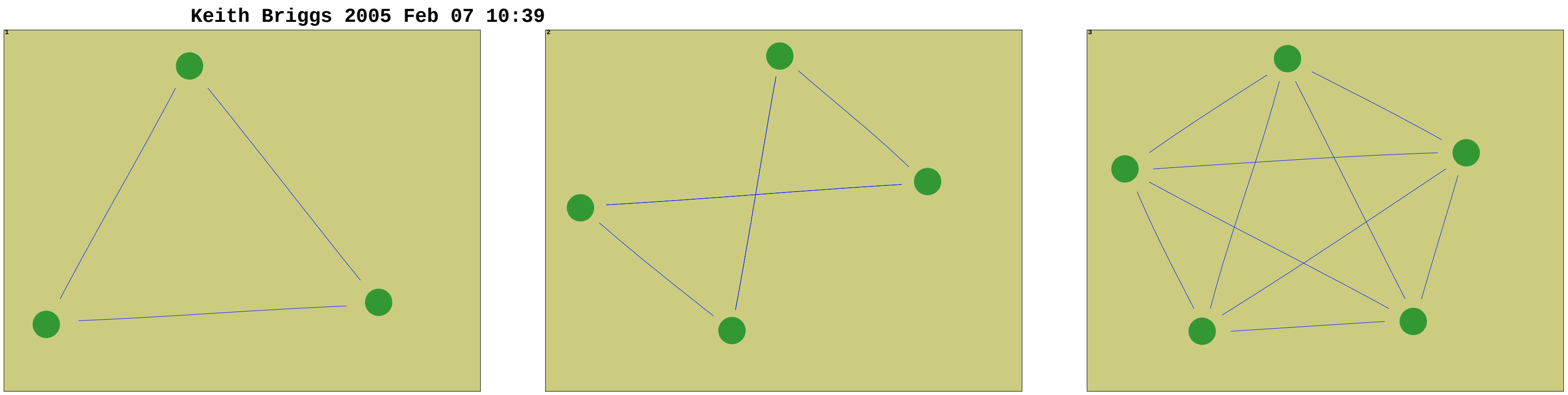
CRG meeting 2005 Feb 08 1500

`planar_graphs.tex` typeset 2005 February 8 12:55 in pdfLaTeX on a linux system

Why?

- we use many graph models in network applications. . .
- many of these are planar - geometric random graphs, Voronoi tesselations, Delaunay triangulations
- until last week, the number of such graphs was unknown
- the breakthrough: Omer Gimenez and Marc Noy
  Asymptotic enumeration and limit laws of planar graphs
  `http://front.math.ucdavis.edu/math.CO/0501269`

Planar and non-planar graphs



the first two are planar; the last one is K_5, which is non-planar

Asymptotics

- Gimenez and Noy: the number of labelled planar graphs on n nodes $p(n)$ has the asymptotic expansion $p(n) \sim g\, n^{-7/2} \gamma^n\, n!$, where $g = 0.0000049700043999\ldots$ and $\gamma = 27.2268777685\ldots$
- I computed the constants more accurately:
  $g = 0.0000049700439997364454985935403006359701128131767643018$
  $\gamma = 27.22687776858857646707945805149445828748980158778683601 0$

Exact enumeration to $n = 42$ by KMB

1	1
2	2
3	8
4	64
5	1023
6	32071
7	1823707
8	163947848
9	20402420291
10	3209997749284
11	604611323732576
12	131861300077834966
13	32577569614176693919
14	8977083127683999891824
15	2726955513946123452637877
16	904755724004585279250537376
17	325403988657293080813790670641
18	126073204858661604803062210068760
19	52337446392225646456388071611054472
20	23173663278385917921578078135542450016
21	10900050834847389498380003477804724863051
22	5427278499581079958628733813434681622714112
23	2851649064665395904628061224490157441123919491
24	1576723196867549751500779544784427710324526483240
25	915097615669129342968080453084386389180260618856043
26	556216030952544916004587932157398727683741873221395376
27	353333727257504600057466229796503418311341496505293976960
28	234137843542082737808316708773268773915428004869540985997760
29	161566245098677397211136945051699885787034376190391470595750059
30	115912331506791061748283633736090378496743635016355705329108763464
31	86331464461316457757237762658675519332157735181905715200428790482261

Exact enumeration to $n=42$ by KMB cotd.

32	66661727121189772175180067570944006070360958404988803103568557089828608
33	5329685035758477481766147047316908966613593259759752702508553960326188374 5
34	440688039494932986465289128101344086154950264598767078315294233849943280088 32
35	37643225028941463223589220761776526017444348648727110614318534777315425022746312
36	3318332904458403754052968414866069632795904280047923636921001095296396028660391481 6
37	301585279350135915261373091984959426530439899714590425122616665974829630414970768540 39
38	28233330443044512277633079717837039450773636582341472184821941285721767718636297903972352
39	2720217447118209880286210430432286955397379024244740247117297188306191617242671627954629905 1
40	269515336136596025091152675854129082131061834814521842827763507402192680131420618754457119093 76
41	27439097133412646381704608313976902021620070544074218052653135888342763410670972063809258349452851
42	2868458043339329234353039414503141778133269460076595979095376699863305126954917063554474982892877734 4

Better asymptotics by fitting to exact data

- Gimenez and Noy's formula is about 6% too small at $n = 42$
- $\frac{p(n)}{n^{-7/2}\,\gamma^n\,n!\,g} \sim 1 - \frac{0.4968}{n} + \frac{153}{n^2} + \dots$
- this is good to about 1% past $n = 10$
- much more could be done here

Connected labelled planar graphs

- egf: $c(x) = \log(1+p(x))$ where $p(x) = \sum_{n=1}^{\infty} p(n)x^n/n!$
- we get $\lim_{n\to\infty} \frac{c(n)}{p(n)} = 0.9632528\ldots$
- at $n = 42$ the fraction is $0.9597031549\ldots$

Connected labelled planar graphs - exact enumeration

1	1
2	1
3	4
4	38
5	727
6	26013
7	1597690
8	149248656
9	18919743219
10	3005354096360
11	569226803220234
12	124594074249852576
13	30861014504270954737
14	8520443838646833231236
15	2592150684565935977152860
16	861079753184429687852978432
17	310008316267496041749182487881
18	120210565158574034465039064701904
19	49939952131087942643302693387881088
20	22126028398071486638215013674791487360
21	10413004988136843726037552067202958282671
22	5187286261027998017964950906419186660388624
23	2726724621370420805637704593781031273318149542
24	1508231586691619168971694492213829810400419586560
25	875650160392350345362890633043411049327995717873275
26	532406079466402052289645363771166588067517135270240768
27	338305081820446777434548193790079455583424626350091319198
28	224237481888925723296995364518205944229164101873982882799104
29	154771497073062185148276613901579968608734935590820735491051129
30	111061995456503299475317908785020231242018947593483292156804735040
31	82735728338664726684277624449644991552102543874870767202897293721744
32	63897253866482099387153875738838812872153253657602612639141236928978944

Exact enumeration cotd.

33	51095541940435199467254291631137210958477914129953550639595248968399841681
34	42255510182452640077564168959832121635573990289216911790064177339358327925504
35	36099803579918572899217704793605296050615609814554407606903578642736859911255260
36	31827289843929444338312132881912913525492073378661899529434559221691835902765135872
37	28929950725087453922829755039067861291837696596575464886510447029548288521794931648999
38	27086570496129997151683406894837226667832603625981713916707833551811837859541651708884224
39	26100372281304868592695927496805611452306290645470949129802423197374955605286815479630114674
40	25862757015603780309015617181334249465222595796515478288841276906171280169844510340444125061120
41	26333388086786970386783733099473556688661058713099244508013345357525697975702485446414342626453571