Graph eigenvalues and connectivity

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2003 July 07 1500

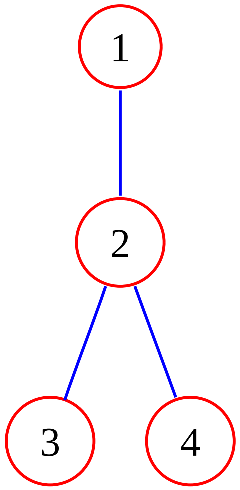
TYPESET 2003 JULY 8 9:13 IN PDFLATEX ON A LINUX SYSTEM

Adjacency matrix

Let Γ be an arbitrary graph with n nodes

Let A be the adjacency matrix of Γ

Example:



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Determining connectivity

Let B be the adjacency matrix with all zeroes replaced by ∞

- for $k = 1, \ldots, n$
  - for $i = 1, \ldots, n$
    - for $j = i+1, \ldots, n$
      - $b_{ij} = \min(b_{ik}+b_{kj}, b_{ij})$

The graph is connected iff all elements b_{ij} $(i \neq j)$ are $< \infty$

We would like to use this for very large graphs, but it takes time $\mathcal{O}(n^3)$ and space $\mathcal{O}(n^2)$!

Graph eigenvalues

Let Γ be an arbitrary graph with n nodes

Let A be the adjacency matrix of Γ

Let Δ be the diagonal matrix with Δ_{ii} the degree of node i

Let $Q \equiv \Delta - A$ be the Laplacian matrix

Let J be the matrix of all ones

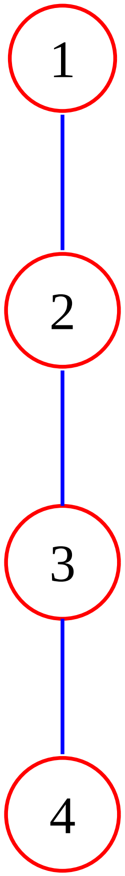
Then the number of spanning trees of Γ is $\kappa = \det(J+Q)/n^2$

Let the spectrum of Q be $0 = \mu_0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{n-1}$

Then we also have $n\kappa = \prod_{i=1}^{n-1} \mu_i$

Thus Γ is connected iff $\mu_1 > 0$

Example 1



$$Q = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

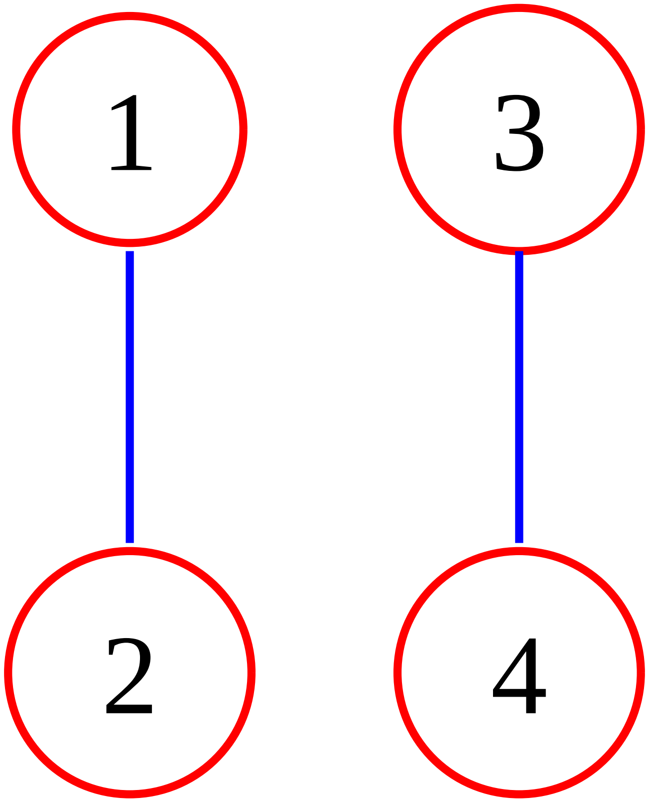
$$\det(J+Q) = 16$$

$$\mu = [0, 0.5858, 2, 3.4142]$$

$\kappa = 1$ from determinant formula

$\kappa = 1$ from eigenvalue formula

Example 2



$$Q = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

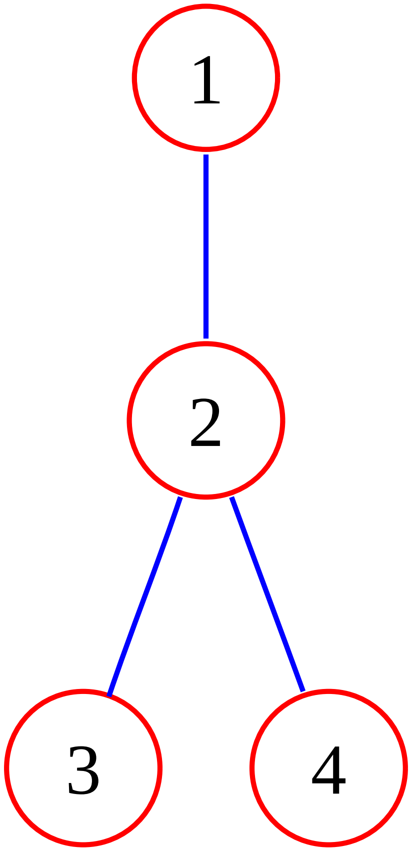
$$\det(J+Q) = 0$$

$$\mu = [0, 0, 2, 2]$$

$\kappa = 0$ from determinant formula

$\kappa = 0$ from eigenvalue formula

Example 3



$$Q = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

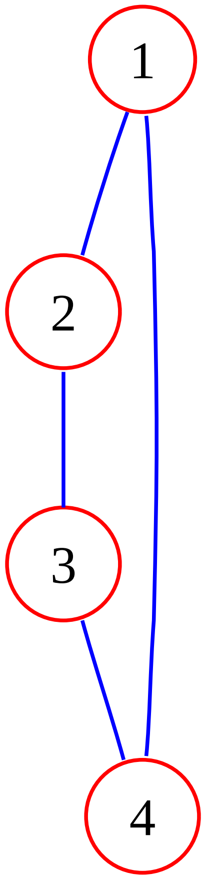
$$\det(J+Q) = 16$$

$$\mu = [0, 1, 1, 4]$$

$\kappa = 1$ from determinant formula

$\kappa = 1$ from eigenvalue formula

Example 4



$$Q = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$\det(J+Q) = 64$

$\mu = [0, 2, 2, 4]$

$\kappa = 4$ from determinant formula

$\kappa = 4$ from eigenvalue formula

My new idea! (2003 July 02)

If $\mu_1 > 0$, the graph is connected, so we need to find only the two smallest eigenvalues, and bound μ_1 away from zero

We can find the two smallest eigenvalues by an inverse QR iteration method: let there be n nodes, let Y_0 by an $n \times 2$ matrix with a 2×2 identity matrix at the top, then iterate for $k = 1, 2, 3, \ldots$

- $QZ = Y_{k-1}$ (solve for Z)
- $Z_k = Y_k R_k$ (QR factorization)

this should work because:

- we can use a sparse representation for Q, and solve for Z with a sparse iterative technique [see references];
- the QR factorization will be very fast for a 2-column matrix

thus, the element R_{22} will converge to the desired μ_1

Problems!

Q is singular. But we can instead use $Q' = J+Q$, where J is a matrix of all ones. The eigenvalues of Q' are $n, \mu_1, \mu_2, \ldots,$ so we now have: $\boxed{G \text{ is connected iff } Q' \text{ is nonsingular}}$

Equivalently, G is connected iff Q' is positive definite.

Inverse QR now works, but probably better methods are available:

- Lanczos iteration (tridiagonalization)
- Arnoldi iteration (ARPACK++)
- SuperLU

the space requirement is now $\mathcal{O}(n)$

the time requirement is now $\mathcal{O}(n^2)$?

challenge:

$\boxed{\text{what is the fastest way to determine whether } Q' \text{ is singular?}}$

References

N. Biggs, *Algebraic graph theory*, CUP 1993

B. Bollobás, *Modern graph theory*, Springer-Verlag 2002

R. Diestel, *Graph theory*, Springer-Verlag 2000

sparse-blas: `http://www.netlib.org/sparse-blas/index.html`

sparse 1.3: `http://www.netlib.org/sparse/index.html`

other sparse codes: `http://gams.nist.gov/serve.cgi/Class/D2a4/`