Some notes on geometric random graphs

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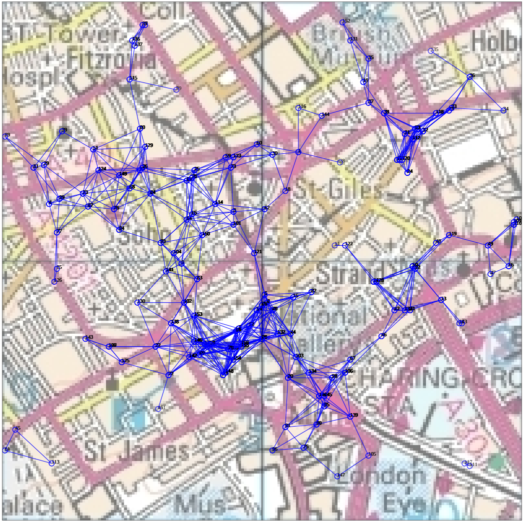
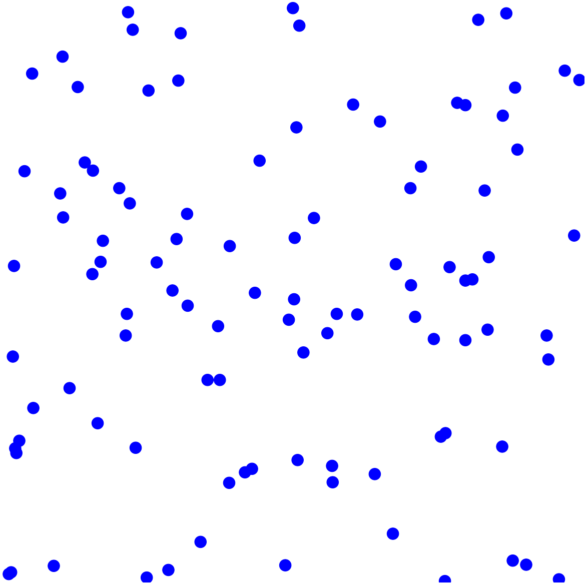
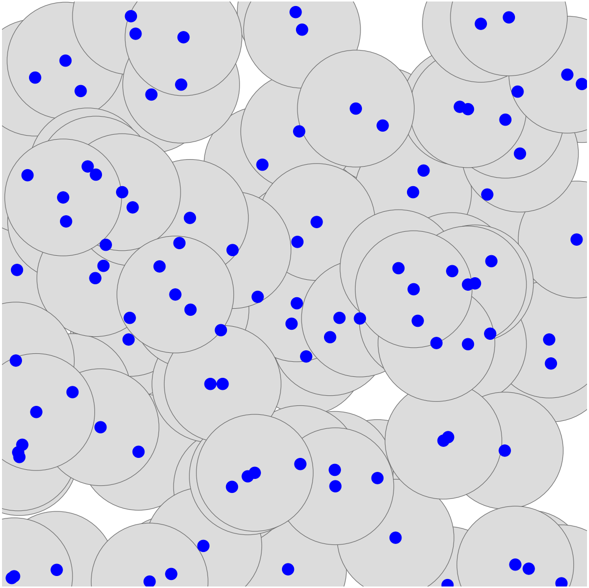
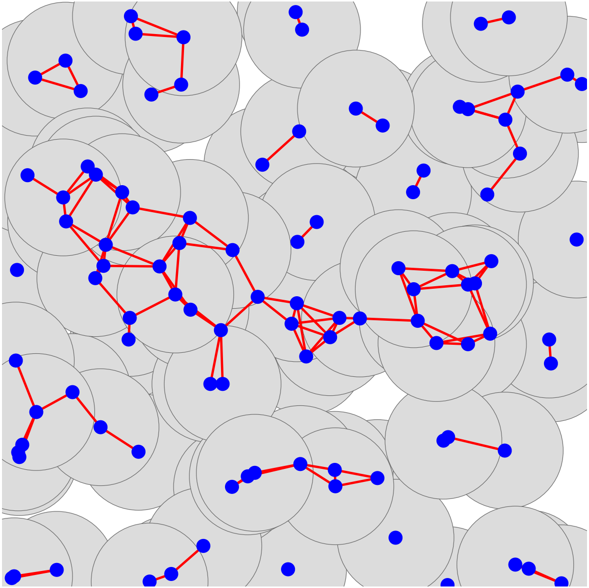
Amorph meeting, University of Sheffield 2008 Sep 09

Wireless networks









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- also binomial PP
- also nonhomogeneous case

PPPP(λ): radial generation

- $s=0$
- do

  $s \leftarrow s - \log(\text{Uniform}(0,1))$
  $\theta = 2\pi\,\text{Uniform}(0,1)$
  $r = \sqrt{s/(\pi\lambda)}$
  $x = r\cos\theta$
  $y = r\sin\theta$

- while $r <$ desired maximum radius

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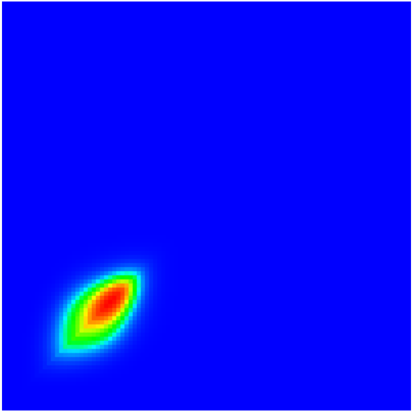
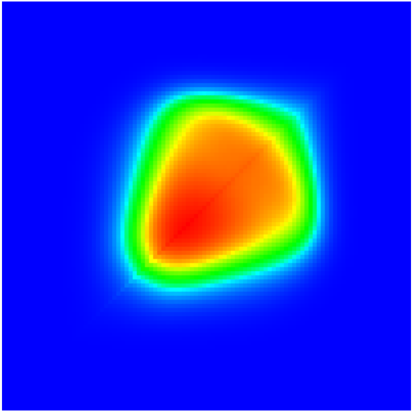
- surprisingly, the degree-degree correlation is the same, independent of λ and ρ !

GRG(20, ρ) degree-degree distribution

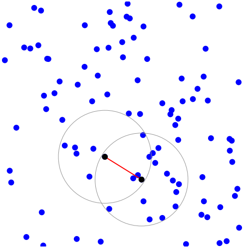


ρ=0.3



ρ=0.5

GRG(λ, ρ) degree-degree correlation



- If $X_0 \sim \text{Poi}(\lambda_0)$, $X_1 \sim \text{Poi}(\lambda_1)$, $X_2 \sim \text{Poi}(\lambda_1)$ are independent, and $Y_1 = X_1 + X_0$, $Y_2 = X_2 + X_0$, then

$$\text{corr}(Y_1, Y_2) = \frac{\lambda_0}{\lambda_0 + \lambda_1}$$

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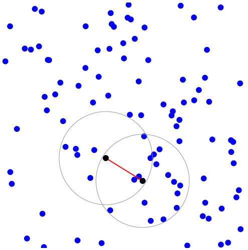

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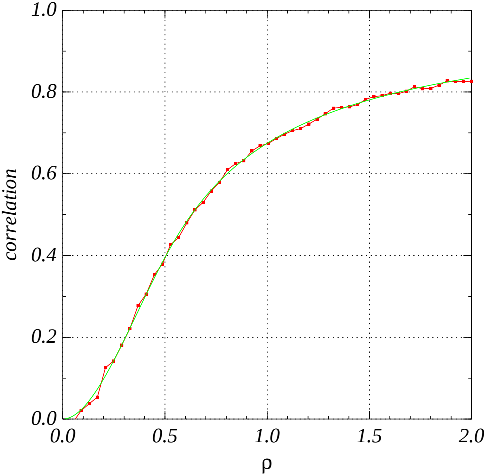
- For PPPP, degree-degree correlation is E[corr]

$$\begin{aligned} &= \int_0^\rho \frac{2\rho^2 \arccos(x/(2\rho)) - (x/2)\sqrt{4\rho^2 - x^2}}{\pi\rho^2} \frac{2x}{\rho^2} \, dx \\ &= 1 - 3\sqrt{3}/(4\pi) \quad \simeq \quad 0.5865 \end{aligned}$$

GRG(λ, ρ) degree-degree correlation - square



exact (doable but messy); simulation

GRG(λ, ρ, unit circle): degree distribution

- pdf of distance of a random point from the centre, given that it is within $1-\rho$ of the edge:

$$f_\rho(x) = \frac{(4-2\rho)x + 2\rho - 2}{\rho^3 - 2\rho^2 + 2\rho} [\![1-\rho < x < 1]\!]$$

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- area of overlap of circles radius 1 and ρ, centres x apart:

$$A(x) = \rho^2 \arccos\left(\frac{x^2+\rho^2-1}{2x\rho}\right) + \arccos\left(\frac{x^2-\rho^2+1}{2x}\right) - \frac{1}{2}[(1-x+\rho)(x+\rho-1)(x-\rho+1)(x+\rho+1)]^{1/2}$$

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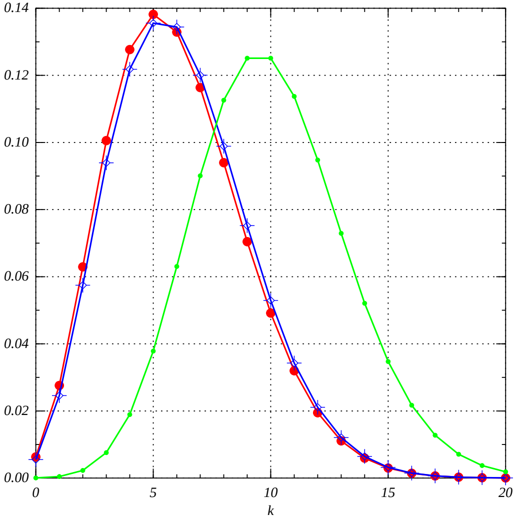
- Prob$[d=k]=$ $(1-\rho)^2\,\text{Poi}(A(x),k)+\rho(2-\rho)\int_{1-\rho}^{1}\text{Poi}(A(x)\lambda)\,f_\rho(x)\text{d}x$
- where $\text{Poi}(\mu,k) = e^{-\mu}\mu^k/k!$

GRG(λ, ρ, unit circle) degree distribution



exact; simulation;
Poisson - ignores edge effect.

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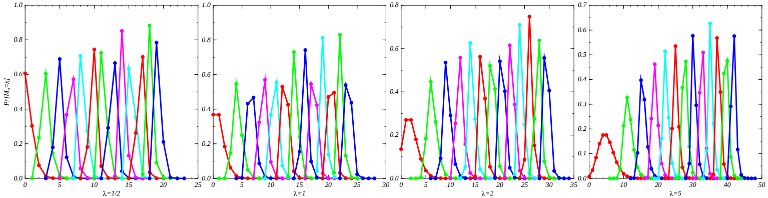

The distribution of the maximum of Poisson variables for $\lambda{=}1/2, 1, 2, 5$ (left to right) and $n{=}10^0, 10^2, 10^4, \ldots, 10^{24}$

Poisson maxima 2

- Anderson: $\exists I_n \in \mathbb{Z}$ s.t. Pr $M_n \in (I_n, I_{n+1}) \to 1$

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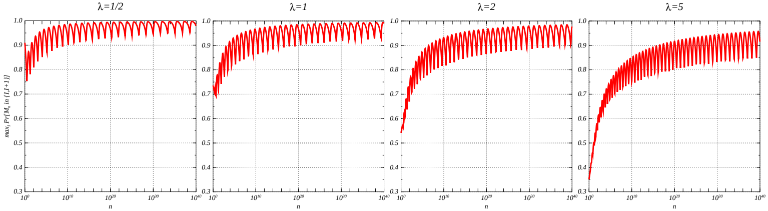

The maximal probability (with respect to I_n) that $M_n \in \{I_n, I_n+1\}$ for $\lambda = 1/2, 1, 2, 5$ (left to right) and $10^0 \leqslant n \leqslant 10^{40}$. The curves show the probability that M_n takes either of its two most frequently occurring values.

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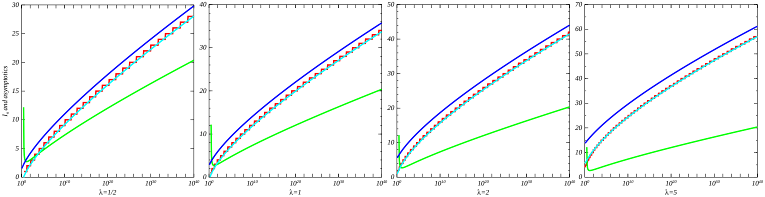
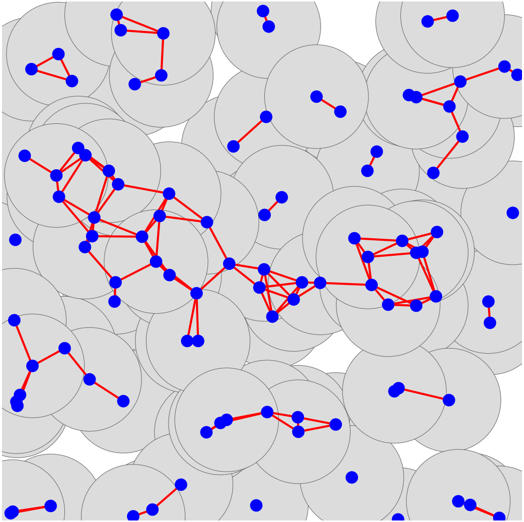


Chromatic number



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